International Journal of Engineering and Applied Physics (IJEAP)
Vol. 2, No. 1, January 2022, pp. 402~412
ISSN: 2737-8071

Uncertainty Principles for the Dunkl-Bessel type transform

Najat Safouane[1], **Radouan Daher**[2]
[1]Department of Mathematics, Faculty of Sciences Aïn Chock, University of Hassan II, Casablanca, Morocco
[2]Department of Mathematics, Faculty of Sciences Aïn Chock, University of Hassan II, Casablanca, Morocco

Article Info

Article history:

Received Sep 20, 2021
Revised Jan 10, 2022
Accepted Sep 20, 2022

Keywords:

Beurling's theorem
Gelfand-Shilov theorem
Cowling-Price's theorem
Morgan's theorem

ABSTRACT

The Dunkl-Bessel type transform satisfies some uncertainty principles similar to the Euclidean Fourier transform. A generalization of Beurling's theorem, Gelfand-Shilov theorem, Cowling-Price's theorem and Morgan's theorem are obtained for the Dunkl-Bessel type transform.

Corresponding Author:

Najat Safouane,
Department of Mathematics,
Faculty of Sciences Aïn Chock, University of Hassan II,Casablanca, Morocco
Email: safouanenajat@live.fr

1. INTRODUCTION AND PRELIMINARIES

There are many theorems known which state that a function and its classical Fourier transform on $\mathbb{R}$ cannot both be sharply localized. That is, it is impossible for a nonzero function and its Fourier transform to be simultaneously small. Here a concept of the smallness had taken different interpretations in different contexts. Hardy [6], Morgan [9], Cowling and Price [5], Beurling [2] for example interpreted the smallness as sharp pointwise estimates or integrable decay of functions. Hardy's theorem [6] for the classical Fourier transform $\mathcal{F}$ on $\mathbb{R}$ asserts that

Theorem 1..1 *Let f be a measurable function on $\mathbb{R}$ such that*

$$|f(x)| \leq Ce^{-ax^2} \quad and \quad |\mathcal{F}(f)(y)| \leq Ce^{-by^2} \tag{1}$$

for some constants $a > 0$, $b > 0$, $C > 0$. We have

- *If $ab > \frac{1}{4}$, then $f = 0$ ae.*
- *If $ab < \frac{1}{4}$, then infinitely nonzero functions satisfy condition (1).*
- *If $ab = \frac{1}{4}$ then $f(x)$*

Considerable attention has been devoted for discovering generalizations to new contexts for the Hardy's theorem. In particular, Cowling and Price [4] have studied an L^p version of Hardy's theorem which states that for $p, q \in [1, \infty]$, at least one of them is finite, if $\|e^{ax^2} f\|_p < \infty$ and $\|e^{by^2} \widehat{f}\|_q < \infty$, then $f = 0$ a.e. if $ab \geq \frac{1}{4}$. . Furthermore, Beurling's theorem, which was found by Beurling and his proof was published much later by Hrmander [7], says that for any non trivial function $f \in L^2(\mathbb{R})$, the product $f(x)\widehat{f}(y)$ is never integrable on $\mathbb{R}^2$ with respect to the measure $e^{|x||y|}dxdy$, where $\widehat{f}$ stands for the Fourier transform of f. A far reaching generalization of this result has been recently proved by Bonami, Demange and Jaming [2]. They proved that if $f \in L^2(\mathbb{R})$ satisfies for an integer N

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)||\widehat{f}(y)|}{(1+|x|+|y|)^N} e^{|x||y|} dxdy < \infty,$$

then f is the form $f(x) = P(x)e^{-bx^2}$, where P is a polynomial of degree strictly lower than $\frac{N-1}{2}$ and b is a positive constant. Morgan [9] has established a famous theorem stating that for $\gamma > 2$ and $\eta = \frac{\gamma}{\gamma-1}$, if $(a\gamma)^{\frac{1}{\gamma}}(b\eta)^{\frac{1}{\eta}} > (\sin(\frac{\pi}{2}(\eta-1)))^{\frac{1}{\eta}}$, $e^{a|x|^\gamma} f \in L^\infty(\mathbb{R})$ and $e^{b|x|^\eta}\mathcal{F}(f) \in L^\infty(\mathbb{R})$. then f is null almost everywhere.

The outline of the content of this paper is as follows. In section 2 we give an analogue of Cowling-Price's theorem for the Dunkl-Bessel type transform $\mathcal{F}_{k,\beta,n}$. Section 3 is devoted to Miyachi's theorem for $\mathcal{F}_{k,\beta,n}$. Section 4 is dedicated to generalize Beurling's theorem for $\mathcal{F}_{k,\beta,n}$. Section 5 is devoted to Morgan's type theorem for $\mathcal{F}_{k,\beta,n}$.

Let us now be more precise and describe our results. To do so, we need to introduce some notations.

Throughout this paper, the letter C indicates a positive constant not necessarily the same in each occurrence. We denote by

- $a_\beta = \dfrac{2\Gamma(\beta+1)}{\sqrt{\pi}\Gamma(\beta+\frac{1}{2})}$, where $\beta > \dfrac{-1}{2}$.
- $x = (x_1, ..., x_{d+1}) = (x^{'}, x_{d+1}) \in \mathbb{R}^d \times]0, \infty[$.
- $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times]0, \infty[$.
- $\lambda = (\lambda_1, ..., \lambda_{d+1}) = (\lambda^{'}, \lambda_{d+1}) \in \mathbb{C}^{d+1}$.
- $C(\mathbb{R}^{d+1})$ the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $E(\mathbb{R}^{d+1})$ (resp. $D(\mathbb{R}^{d+1})$) the space of C^∞ functions on $\mathbb{R}^{d+1}$, even with respect to the last variable (resp. with compact support).
- $\mathcal{R}$ the root system in $\mathbb{R}^d\backslash\{0\}$, $\mathcal{R}_+$ is a fixed positive subsystem and $k \in \mathcal{R} \to]0, \infty[$ a multiplicity function.
- w_k the weight function defined by

$$w_k(x^{'}) = \prod_{\alpha\in\mathcal{R}_+} |<\alpha, x^{'}>|^{2k(\alpha)},\ x^{'} \in \mathbb{R}^d.$$

- $L^p_{k,\beta}(\mathbb{R}^d \times \mathbb{R}_+),\ 1 \le p \le +\infty$ the space of measurable functions on $\mathbb{R}^d \times \mathbb{R}_+$ such that

$$\|f\|_{k,\beta,p} = \left(\int_{\mathbb{R}^d\times\mathbb{R}_+} |f(x)|^p d\mu_{k,\beta}(x)dx\right)^{\frac{1}{p}} < +\infty,\ if\ 1 \le p < +\infty, \tag{2}$$

$$\|f\|_{k,\beta,\infty} = ess \sup_{x\in\mathbb{R}^d\times[0,+\infty[} |f(x)| < +\infty,\ if\ p = \infty \tag{3}$$

where

$$\mu_{k,\beta}(x)dx = w_k(x')x^{2\beta+1}_{d+1}dx'dx_{d+1},\ \ x = (x', x_{d+1}) \in \mathbb{R}^d \times \mathbb{R}^+. \tag{4}$$

- $\mathcal{M}_n$ the map defined by $\mathcal{M}_n f(x^{'}, x_{d+1}) = x^{2n}_{d+1} f(x^{'}, x_{d+1})$.
- $L^p_{k,\beta,n}(\mathbb{R}^{d+1}_+)$ the class of measurable functions f on $\mathbb{R}^{d+1}_+$ for which

$$\|f\|_{k,\beta,n,p} = \|\mathcal{M}_n^{-1} f\|_{k,\beta+2n,p} < \infty.$$

- $E_n(\mathbb{R}^{d+1})$ (resp. $D_n(\mathbb{R}^{d+1})$) stand for the subspace of $E(\mathbb{R}^{d+1})$ (resp. $D(\mathbb{R}^{d+1})$) consisting of functions f such that
$$f(x^{'},0)=\left(\frac{d^kf}{dx_{d+1}^k}\right)(x^{'},0)=0,\ \forall k\in\{1,...2n-1\}.$$

In this section we recall some facts about harmonic analysis related to the Dunkl-Bessel type operator $\mathcal{F}_{k,\beta,n}$. We cite here, as briefly as possible, only some properties. For more details we refer to [1].

Definition 1..2 *For all $x\in\mathbb{R}^d\times]0,\infty[$ we define the measure $\xi_x^{k,\beta}$ on $\mathbb{R}^d\times]0,\infty[$ by*
$$d\xi_x^{k,\beta}(y)=a_\beta x_{d+1}^{-2\beta}(x_{d+1}^2-y_{d+1}^{d+1})^{\beta-\frac{1}{2}}1_{]0,x_{d+1}[}(y_{d+1})d\mu_{x'}(y^{'})dy_{d+1},$$
where $\mu_{x'}$ is a probability measure on $\mathbb{R}^d$, with support in the closed ball $B(o,\|x\|)$ of center o and radius $\|x\|$. $1_{]0,x_{d+1}[}$ is the characteristic function of the interval $]0,x_{d+1}[$.

For all $y\in\mathbb{R}^d$, we define the measure $\varrho_y^{k,\beta}$ on $\mathbb{R}^d\times[0,\infty[$, by
$$d\varrho_y^{k,\beta}(x)=a_\beta(x_{d+1}^2-y_{d+1}^2)^{\beta-\frac{1}{2}}x_{d+1}1_{]y_{d+1},\infty[}(x_{d+1})d\nu_{y'}(x^{'})dx_{d+1}. \quad (5)$$

We define the heat functions $W_{s,p}^{k,\beta}(r,.)$ related to the Dunkl-Bessel type Laplacian $\Delta_{k,\beta,n}$ by
$$\forall y\in\mathbb{R}_+^{d+1},\ W_{s,p}^{k,\beta}(r,y)=\frac{i^{|s|}(-1)^pc_k^2}{4^{\gamma+\beta+d}(\Gamma(\beta+1))^2}y_{d+1}^{2n}\int_{\mathbb{R}_+^{d+1}}x_1^{s_1}...x_d^{s_d}x_{d+1}^{2p}e^{-r\|x\|^2}\Lambda(x,y)d\mu_{k,\beta+n}(x). \quad (6)$$

These functions satisfy the following properties
$$\forall y\in\mathbb{R}_+^{d+1},\ \mathcal{F}_{k,\beta,n}(W_{s,p}^{k,\beta}(r,.))(y)=i^{|s|}(-1)^py_1^{s_1}...y_d^{s_d}y_{d+1}^{2p}e^{-r\|y\|^2} \quad (7)$$

Definition 1..3 *The Dunkl-Bessel type intertwining operator is the operator $\mathcal{R}_{k,\beta,n}$ defined on $C(\mathbb{R}^{d+1})$ by*
$$\mathcal{R}_{k,\beta,n}f(x)=\int_{\mathbb{R}^{d+1}}x_{d+1}^{2n}f(y)d\xi_x^{k,\beta+2n}(y).$$

Definition 1..4 *The dual of the Dunkl-Bessel type intertwining operator $\mathcal{R}_{k,\beta,n}$ is the operator defined on $D_n(\mathbb{R}^{d+1})$ by: $\forall y=(y^{'},y_{d+1})\in\mathbb{R}^d\times]0;\infty[$,*
$${}^t\mathcal{R}_{k,\beta,n}(f)(y)=\int_{\mathbb{R}_+^{d+1}}x_{d+1}^{-2n}f(x)d\varrho_y^{k,\beta+2n}(x). \quad (8)$$

Proposition 1..5 *Let f be in $L^1_{k,\beta,n}(\mathbb{R}_+^{d+1})$. Then*
$$\int_{\mathbb{R}_+^{d+1}}{}^t\mathcal{R}_{k,\beta,n}(f)(y)dy=\int_{\mathbb{R}_+^{d+1}}f(x)d\mu_{k,\beta+n}(x)dx.$$

Theorem 1..6 *Let $f\in L^1_{k,\beta,n}(\mathbb{R}_+^{d+1})$ and $g\in C(\mathbb{R}^{d+1})$, we have ${}^t\mathcal{R}_{k,\beta,n}(f)$ is defined almost every where on $\mathbb{R}_+^{d+1}$ and the following formula*
$$\int_{\mathbb{R}_+^{d+1}}{}^t\mathcal{R}_{k,\beta,n}(f)(y)g(y)dy=\int_{\mathbb{R}_+^{d+1}}f(x)\mathcal{R}_{k,\beta,n}(g)(x)d\mu_{k,\beta+n}(x)dx.$$

We consider the function $\Lambda_{k,\beta,n}$, given for $\lambda=(\lambda',\lambda_{d+1})\in\mathbb{C}^d\times\mathbb{C}$ by
$$\Lambda_{k,\beta,n}(x,\lambda)=x_{d+1}^{2n}K(x',-i\lambda')j_\beta(x_{d+1}\lambda_{d+1}), \quad (9)$$
where $j_\beta(x_{d+1}\lambda_{d+1})$ is the normalized Bessel function defined by
$$j_\beta(x_{d+1}\lambda_{d+1})=a_\beta\int_0^1(1-t^2)^{\beta-\frac{1}{2}}\cos(x_{d+1}\lambda_{d+1}t)dt$$

and $K(x', -i\lambda')$ is the Dunkl Kernel defined by

$$K(x', -i\lambda') = \int_{\mathbb{R}^d} e^{-i<y,\lambda'>} d\mu_{x'}(y).$$

Definition 1..7 *The Dunkl-Bessel type transform is given for f in $D_n(\mathbb{R}^{d+1})$ by*

$$\forall \lambda \in \mathbb{R}^d \times \mathbb{R}_+, \ \mathcal{F}_{k,\beta,n}(f)(\lambda) = \int_{\mathbb{R}^d \times \mathbb{R}_+} f(x)\Lambda_{k,\beta,n}(x,\lambda) d\mu_{k,\beta}(x)dx. \tag{10}$$

Proposition 1..8 *For $f \in D_n(\mathbb{R}^{d+1})$, we have*

$$\mathcal{F}_{k,\beta,n}(f) = \mathcal{F}_0 \circ {}^t\mathcal{R}_{k,\beta,n}(f), \tag{11}$$

where $\mathcal{F}_0$ is the transform defined by $\forall \lambda = (\lambda', \lambda_{d+1}) \in \mathbb{R}^d \times \mathbb{R}_+$

$$\mathcal{F}_0(f)(\lambda', \lambda_{d+1}) = \int_{\mathbb{R}^d \times \mathbb{R}_+} f(x', x_{d+1}) e^{-i<\lambda', x_{d+1}>} \cos(x_{d+1}\lambda_{d+1}) dx' dx_{d+1}.$$

We denote by $L^p_{k,\beta,n}(\mathbb{R}^{d+1}_+)$ the class of measurable functions f on $\mathbb{R}^{d+1}_+$ for which

$$\|f\|_{k,\beta,n,p} = \|\mathcal{M}_n^{-1} f\|_{k,\beta+2n,p} < \infty. \tag{12}$$

2. BEURLING'S THEOREM FOR THE DUNKL-BESSEL TYPE TRANSFORM

To prove the main theorem of this section we need the following lemmas.

Lemma 2..1 *Let $N \geq 0$. We consider f in $L^2_{k,\beta,n}(\mathbb{R}^{d+1}_+)$ satisfying*

$$\int_{\mathbb{R}^{d+1}_+} \int_{\mathbb{R}^{d+1}_+} \frac{|f(x)||\mathcal{F}_{k,\beta,n}(f)(y)|}{(1 + \|x\| + \|y\|)^N} e^{\|x\|\|y\|} d\mu_{k,\beta+n}(x) dy < +\infty. \tag{13}$$

Then $f \in L^1_{k,\beta,n}(\mathbb{R}^{d+1}_+)$.

Proof. Using the Fubini's theorem and the relation (13) we have for almost every $y \in \mathbb{R}^{d+1}_+$:

$$\frac{|\mathcal{F}_{k,\beta,n}(f)(y)|}{(1 + \|y\|)^N} \int_{\mathbb{R}^{d+1}_+} \frac{|f(x)|}{(1 + \|x\|)^N} e^{\|x\|\|y\|} d\mu_{k,\beta+n}(x) < +\infty.$$

As f is not negligible, there exists $y_0 \in \mathbb{R}^{d+1}_+$, $y_0 \neq 0$ such that $\mathcal{F}_{k,\beta,n}(f)(y_0) \neq 0$. Thus

$$\int_{\mathbb{R}^{d+1}_+} \frac{|f(x)|}{(1 + \|x\|)^N} e^{\|x\|\|y_0\|} d\mu_{k,\beta+n}(x) < +\infty. \tag{14}$$

Since the function $\dfrac{e^{\|x\|\|y_0\|}}{(1 + \|x\|)^N}$ is greater than 1 for large $\|x\|$, then

$$\begin{aligned}
\int_{\mathbb{R}^{d+1}_+} |f(x)| d\mu_{k,\beta+n}(x) &< +\infty \\
\int_{\mathbb{R}^{d+1}_+} \frac{|f(x)|}{x_{d+1}^{2n}} d\mu_{k,\beta+2n}(x) &< +\infty \\
\int_{\mathbb{R}^{d+1}_+} |\mathcal{M}_n^{-1} f(x)| d\mu_{k,\beta+2n}(x) &< +\infty
\end{aligned}$$

which proves that $f \in L^1_{k,\beta,n}(\mathbb{R}^{d+1}_+)$. ■

Theorem 2..2 *Let $N \in \mathbb{N}$ and $f \in L^2_{k,\beta,n}(\mathbb{R}^{d+1}_+)$ satisfying (13). Then*

- *If $N \geq d+2$ we have*
$$f(y) = \sum_{|s|+p<\frac{N-d-1}{2}} a^{k,\beta}_{s,p} W^{k,\beta}_{s,p}(r,y),\ y \in \mathbb{R}^{d+1}_+, \tag{15}$$
where $r > 0,\ a^{k,\beta}_{s,p} \in \mathbb{C}$ and $W^{k,\beta}_{s,p}(r,.)$ given by the relation (6).

- *Else $f(y) = 0$ a.e $y \in \mathbb{R}^{d+1}_+$.*

Proof. From Lemma 1 and Theorem 2, the function f belongs to $L^1_{k,\beta,n}(\mathbb{R}^{d+1}_+)$ and the function ${}^tR_{k,\beta,n}(f)$ is defined almost everywhere on $\mathbb{R}^{d+1}_+$. We shall prove that

$$\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \frac{e^{\|x\|\|y\|}|{}^t\mathcal{R}_{k,\beta,n}f(x)||\mathcal{F}_0({}^t\mathcal{R}_{k,\beta,n})(y)|}{(1+\|x\|+\|y\|)^N} dydx < +\infty. \tag{16}$$

Take y_0 as in Lemma 1. We write the above integral as a sum of the following integrals

$$I = \int_{\mathbb{R}^{d+1}_+}\int_{\|y\|\leq\|y_0\|} \frac{e^{\|x\|\|y\|}}{(1+\|x\|+\|y\|)^N}|{}^t\mathcal{R}_{k,\beta,n}f(x)||\mathcal{F}_0({}^t\mathcal{R}_{k,\beta}(f))(y)|dydx$$

and

$$J = \int_{\mathbb{R}^{d+1}_+}\int_{\|y\|\geq\|y_0\|} \frac{e^{\|x\|\|y\|}}{(1+\|x\|+\|y\|)^N}|{}^t\mathcal{R}_{k,\beta,n}f(x)||\mathcal{F}_0({}^t\mathcal{R}_{k,\beta,n}(f))(y)|dydx.$$

We will prove that I and J are finite, which implies (16).

- As the functions $|\mathcal{F}_{k,\beta,n}(f)(y)|$ is continuous in the compact $y \in \mathbb{R}^{d+1}_+/\|y\| \leq \|y_0\|$, so we get
$$I \leq C\int_{\mathbb{R}^{d+1}_+} \frac{e^{\|x\|\|y_0\|}|{}^t\mathcal{R}_{k,\beta,n}f(x)|}{(1+\|x\|)^N}dx.$$
Writing the integral of the second member as $I_1 + I_2$ with
$$I_1 = \int_{\|x\|\leq\frac{N}{\|y_0\|}} \frac{e^{\|x\|\|y_0\|}|{}^t\mathcal{R}_{k,\beta,n}f(x)|}{(1+\|x\|)^N}dx$$
and
$$I_2 = \int_{\|x\|\geq\frac{N}{\|y_0\|}} \frac{e^{\|x\|\|y_0\|}|{}^t\mathcal{R}_{k,\beta,n}f(x)|}{(1+\|x\|)^N}dx.$$
There for, we have the following results:
  - As the function $x \to \dfrac{e^{\|x\|\|y_0\|}}{(1+\|x\|)^N}$ is continuous in the compact $x \in \mathbb{R}^{d+1}_+/\|x\| \leq \frac{N}{\|y_0\|}$, and f is in $L^1_{k,\beta,n}(\mathbb{R}^{d+1}_+)$ we deduce by using Fubini-Tonelli's theorem, and the relation (5), (7) that ${}^t\mathcal{R}_{k,\beta,n}(|f|)$ belong to $L^1_{k,\beta,n}(\mathbb{R}^{d+1}_+)$. Hence I_1 is finite.
  - On the other hand, for $t > \frac{N}{\|y_0\|}$, the function $t \to \dfrac{e^{t\|y_0\|}}{(1+t)^N}$ is increasing, so we obtain by using Fubini-Tonelli's theorem, and (5), (8) and Proposition 1, that
  $$I_2 \leq \int_{\mathbb{R}^{d+1}_+} \frac{e^{\|\xi\|\|y_0\|}}{(1+\|\xi\|)^N}|f(\xi)|d\mu_{k,\beta+n}(\xi).$$
  The inequality (14) assert that I_2 is finite. This proves that I is finite.

- We suppose $\|y_0\| \leq N$. Let $J = J_1 + J_2 + J_3$, with

$$J_1 = \int_{\|x\| \leq \frac{N}{\|y_0\|}} \int_{\|y_0\| \leq \|y\| \leq N} \frac{e^{\|x\|\|y\|}}{(1+\|x\|+\|y\|)^N} |{}^t\mathcal{R}_{k,\beta,n}(f)(x)||\mathcal{F}_{k,\beta,n}(f)(y)|dydx.$$

$$J_2 = \int_{\|x\| \geq \frac{N}{\|y_0\|}} \int_{\|y_0\| \leq \|y\| \leq N} \frac{e^{\|x\|\|y\|}}{(1+\|x\|+\|y\|)^N} |{}^t\mathcal{R}_{k,\beta,n}(f)(x)||\mathcal{F}_{k,\beta,n}(f)(y)|dydx.$$

$$J_3 = \int_{\mathbb{R}_+^{d+1}} \int_{\|y\| \geq N} \frac{e^{\|x\|\|y\|}}{(1+\|x\|+\|y\|)^N} |{}^t\mathcal{R}_{k,\beta,n}(f)(x)||\mathcal{F}_{k,\beta,n}(f)(y)|dydx.$$

  - As the function $(x,y) \rightarrow \dfrac{e^{\|x\|\|y\|}}{(1+\|x\|+\|y\|)^N}|\mathcal{F}_{k,\beta,n}(f)(y)|$ is bounded in the compact $\{x \in \mathbb{R}_+^{d+1}/\|x\| \leq \frac{N}{\|y_0\|}\} \times \{\xi \in \mathbb{R}_+^{d+1}/\|y_0\| \leq \|\xi\| \leq N\}$ and ${}^t\mathcal{R}_{k,\beta,n}(|f|)(x)$ is Lebesgue-integrable on $\mathbb{R}_+^{d+1}$, then J_1 is finite.
  - Let $\lambda > 0$. As the function $t \rightarrow \dfrac{e^{\lambda t}}{(1+t+\lambda)^N}$ is increasing for $t > \frac{N}{\lambda}$. Thus, for all $(x,y) \in C(\xi, y_0, N)$ we have the inequality

$$\frac{e^{\|x\|\|y\|}}{(1+\|x\|+\|y\|)^N} \leq \frac{e^{\|\xi\|\|y\|}}{(1+\|\xi\|+\|y\|)^N},$$

  with $C(\xi, y_0, N) = \{(x,y) \in \mathbb{R}_+^{d+1} \times \mathbb{R}_+^{d+1} / \frac{N}{\|y\|} \leq \|x\| \leq \|\xi\| \; and \; \|y_0\| \leq \|y\| \leq N\}$. Therefore, from Fubini-Tonelli's theorem and the relations (5), (8), we get

$$J_2 \leq \int_{\mathbb{R}_+^{d+1}} \int_{\mathbb{R}_+^{d+1}} |(f)(\xi)||\mathcal{F}_{k,\beta,n}(f)(y)| \frac{e^{\|\xi\|\|y\|}}{(1+\|\xi\|+\|y\|)^N} dy d\mu_{k,\beta+n}(\xi).$$

  Taking account of the condition (13), we deduce that J_2 is finite.
  - For $\|y\| > N$, the function $t \rightarrow \dfrac{e^{t\|y\|}}{(1+t+\|y\|)^N}$ is increasing. We deduce, by using Fubini-Tonelli's theorem and the relations (5), (8), (13) that

$$J_3 \leq \int_{\mathbb{R}_+^{d+1}} \int_{\|y\| > N} |(f)(\xi)||F_{k,\beta,n}(f)(y)| \frac{e^{\|\xi\|\|y\|}}{(1+\|\xi\|+\|y\|)^N} dy d\mu_{k,\beta+n}(\xi) < +\infty.$$

  This implies that J_3 is finite.
  Finally for $\|y_0\| > N$, we have $J \leq J_3 < \infty$. This completes the proof of the relation (16).

According to Corollary 3.1, ii) of [4], we deduce that

$$\forall x \in \mathbb{R}_+^{d+1}, \; {}^t\mathcal{R}_{k,\beta,n}(f)(x) = P(x)e^{-\delta\|x\|^2}$$

with $\delta > 0$ and P a polynomial of degree strictly lower than $\frac{N-d-1}{2}$.
Using this relation and (6), we deduce that

$$\forall x \in \mathbb{R}_+^{d+1}, \; \mathcal{F}_{k,\beta,n}(f)(y) = \mathcal{F}_0 \circ^t \mathcal{R}_{k,\beta,n}(f)(y) = \mathcal{F}_0(P(x)e^{-\delta\|x\|^2})(y).$$

But

$$\forall x \in \mathbb{R}_+^{d+1}, \; \mathcal{F}_0(P(x)e^{-\delta\|x\|^2})(y) = Q(y)e^{\frac{-\|y\|^2}{4\delta}},$$

With Q a polynomial of degree strictly lower than $\frac{N-d-1}{2}$.
Thus from (7) we obtain

$$\forall x \in \mathbb{R}_+^{d+1}, \; \mathcal{F}_{k,\beta,n}(f)(y) = \mathcal{F}_{k,\beta,n}\left(\sum_{|s|+p<\frac{N-d-1}{2}} a_{s,p}^{k,\beta} W_{s,p}^{k,\beta}(\frac{1}{4\delta},.)\right)(y).$$

The injectivity of the transform $\mathcal{F}_{k,\beta,n}$ implies

$$\forall x \in \mathbb{R}_+^{d+1},\ f(x) = \sum_{|s|+p<\frac{N-d-1}{2}} a_{s,p}^{k,\beta} W_{s,p}^{k,\beta}(\frac{1}{4\delta}, .)(x)\ a.e,$$

and the theorem is proved.

■

3. GELFAND-SHILOV TYPE FOR THE DUNKL-BESSEL TYPE TRANSFORM

In this section we give analogue of the Gelfand-Shilov for the Dunkl-Bessel type transform $\mathcal{F}_{k,\beta,n}$.

Theorem 3..1 *(Gelfand-Shilov type) Let $N \in \mathbb{N}$ and assume that $f \in L^2_{k,\beta}(\mathbb{R}_+^{d+1})$ is such that*

$$\int_{\mathbb{R}_+^{d+1}} \frac{|f(x)|e^{\frac{(2a)^p}{p}\|x\|^p}}{(1+\|x\|)^N} d\mu_{k,\beta+n}(x) < +\infty, \tag{17}$$

$$\int_{\mathbb{R}_+^{d+1}} \frac{|\mathcal{F}_{k,\beta,n}(f)(y)|e^{\frac{(2b)^q}{q}\|y\|^q}}{(1+\|y\|)^N} dy < +\infty \tag{18}$$

Where $1 < p,\ q < +\infty,\ \frac{1}{p} + \frac{1}{q} = 1,\ a > 0,\ b > 0$ and $ab \geq \frac{1}{4}$. Then:

1. *If $ab > \frac{1}{4}$, we have $f(x) = 0$ a.e.*
2. *We suppose that $ab = \frac{1}{4}$.*
   - *If $N < \frac{d}{2} + 1,\ 1 < p, q < +\infty$ we have $f(x) = 0,\ a.e\ x \in \mathbb{R}^d$.*
   - *If $N \geq \frac{d}{2} + 1$.*
     - *For the cases: $2 \leq q < +\infty,\ 1 < p < +\infty$,*
       $1 < q < 2,\ 2 < p < +\infty$,
       $q = 2,\ p = 2$
       we have $f(x) = 0,\ a.e\ x \in \mathbb{R}^d$.
     - *For the case $1 < q < 2,\ 1 < p < 2$,*
       we have
       $$f(x) = \sum_{|s|+p<\frac{2N-d-1}{2}} a_{s,p}^{k,\beta} W_{s,p}^{k,\beta}(r, x),\ a.e.\ x \in \mathbb{R}_+^{d+1}, \tag{19}$$
       Where $r > 0$ and $a_{s,p}^{k,\beta} \in \mathbb{C}$.
     - *For the case $q = 2$, $1 < p < 2$*
       If $0 < r \leq 2b^2$ we have $f(x) = 0,\ a.e\ x \in \mathbb{R}_+^{d+1}$.
       If $r > 2b^2$ the function f is given by the relation (18).
     - *For the case $p = 2$, $1 < q < 2$*
       If $r \geq 2b^2$ we have $f(x) = 0,\ a.e\ x \in \mathbb{R}_+^{d+1}$.
       If $0 < r < 2b^2$ the function f is given by the relation (18).

Proof. Using the inequality

$$4ab\|x\|\|y\| \leq \frac{(2a)^p}{p}\|x\|^p + \frac{(2b)^q}{q}\|y\|^q,$$

we get

$$\int_{\mathbb{R}_+^{d+1}} \int_{\mathbb{R}_+^{d+1}} \frac{|f(x)||\mathcal{F}_{k,\beta,n}(f)(y)|}{(1+\|x\|+\|y\|)^{2N}} e^{4ab\|x\|\|y\|} dy d\mu_{k,\beta+n}(x) \leq$$

$$\int_{\mathbb{R}_+^{d+1}} \frac{|f(x)|e^{\frac{(2a)^p}{p}\|x\|^p}}{(1+\|x\|)^N} d\mu_{k,\beta+n}(x) \int_{\mathbb{R}_+^{d+1}} \frac{|\mathcal{F}_{k,\beta,n}(f)(y)|e^{\frac{(2b)^q}{q}\|y\|^q}}{(1+\|y\|)^N} dy < +\infty. \tag{20}$$

As $ab \geq \frac{1}{4}$, then from (20) we deduce that the condition (14) is satisfied. By using the proof of Theorem 3, we obtain, $\forall x \in \mathbb{R}_+^{d+1}$,

$$ {}^t\mathcal{R}_{k,\beta}(f)(x) = P(x)e^{-\frac{\|x\|^2}{4r}}; \forall x \in \mathbb{R}_+^{d+1}, \mathcal{F}_{k,\beta,n}(f)(y) = Q(y)e^{-r\|y\|^2}, \tag{21}$$

where r is a positive constant and P, Q are polynomials of the same degree which is strictly lower than $\frac{2N-d-1}{2}$.
1) From (20) and the proof of (16) we deduce that

$$\int_{\mathbb{R}_+^{d+1}} \int_{\mathbb{R}_+^{d+1}} \frac{|{}^t\mathcal{R}_{k,\beta} f(x)||\mathcal{F}_0({}^t\mathcal{R}_{k,\beta}(f))(y)|}{(1+\|x\|+\|y\|)^{2N}} e^{4ab\|x\|\|y\|} dxdy < +\infty, \tag{22}$$

By replacing in (22) the functions ${}^t\mathcal{R}_{k,\beta,n}(f)(x)$ and $\mathcal{F}_{k,\beta,n}(f)(y)$ by their expression given in (21), we get

$$\int_{\mathbb{R}_+^{d+1}} \int_{\mathbb{R}_+^{d+1}} \frac{|P(x)||Q(y)|}{(1+\|x\|+\|y\|)^{2N}} e^{-(\sqrt{r}\|y\| - \frac{1}{2\sqrt{r}}\|x\|)^2} e^{(4ab-1)\|x\|\|y\|} dxdy < +\infty, \tag{23}$$

As $ab > \frac{1}{4}$, there exists $\varepsilon > 0$ such that $4ab - 1 - \varepsilon > 0$. If P is non null, Q is also non null and we have

$$\int_{\mathbb{R}_+^{d+1}} \int_{\mathbb{R}_+^{d+1}} \frac{|P(x)||Q(y)|}{(1+\|x\|+\|y\|)^{2N}} e^{-(\sqrt{r}\|y\| - \frac{1}{2\sqrt{r}}\|x\|)^2} e^{(4ab-1)\|x\|\|y\|} dxdy$$
$$\geq C \int_{\mathbb{R}_+^{d+1}} \int_{\mathbb{R}_+^{d+1}} e^{-(\sqrt{r}\|y\| - \frac{1}{2\sqrt{r}}\|x\|)^2} e^{(4ab-1-\varepsilon)\|x\|\|y\|} dxdy,$$

Where C is a positive constant. But the function

$$e^{-(\sqrt{r}\|y\| - \frac{1}{2\sqrt{r}}\|x\|)^2} e^{(4ab-1-\varepsilon)\|x\|\|y\|}$$

is not integrable, (23) does not hold. Hence $f(x) = 0$ a.e.
2)
i) We deduce the result from (20) and Theorem 3.
ii) By using (20) the relations (9), (11) can also be written in the form

$$\int_{\mathbb{R}^d} \frac{|\mathcal{F}_{k,\beta,n}(f)(y)|e^{\frac{(2b)^q}{q}\|y\|^q}}{(1+\|y\|)^N} dy = \int_{\mathbb{R}^d} \frac{|Q(y)|e^{-r\|y\|^2} e^{\frac{(2b)^q}{q}\|y\|^q}}{(1+\|y\|)^N} dy.$$

and

$$\int_{\mathbb{R}^d} \frac{|f(x)|e^{\frac{(2a)^p}{p}\|x\|^p}}{(1+\|x\|)^N} \omega_k(x)dx = \int_{\mathbb{R}^d} \frac{|P(x)|e^{-\frac{\|x\|^2}{4r}} e^{\frac{(2a)^p}{p}\|x\|^p}}{(1+\|x\|)^N} \omega_k(x)dx.$$

We obtain ii) from Theorem 3 and by studying the convergence of these integrals as we have made it in 1). ■

4. HARDY TYPE FOR THE DUNKL-BESSEL TYPE TRANSFORM

Theorem 4..1 *(Hardy type) Let $N \in \mathbb{N}$. Assume that $f \in L^2_{k,\beta}(\mathbb{R}_+^{d+1})$ is such that*

$$|f(x)| \leq Me^{-\frac{1}{4a}\|x\|^2} \ a.e$$

and

$$\forall x \in \mathbb{R}_+^{d+1}, |\mathcal{F}_{k,\beta,n}(f)(y)| \leq M(1+|y_j|)^N e^{-b|y_j|^2}, j = 1, ..., d+1, \tag{24}$$

for some constants $a > 0$, $b > 0$ and $M > 0$. Then,
i) If $ab > \frac{1}{4}$, then $f = 0$ a.e.
ii) If $ab = \frac{1}{4}$, the function f is of the form

$$f(x) = \sum_{|s|+p \leq N} a_{s,p}^{k,\beta} W_{s,p}^{k,\beta}(\frac{1}{4a}, x) \ a.e. \ where \ a_{s,p}^{k,\beta} \in \mathbb{C}.$$

iii) If $ab < \frac{1}{4}$, there are infinity many nonzero functions f satisfying the condition (24).

Proof. The first condition of (24) implies that $f \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$. So by Theorem 2, the function ${}^t\mathcal{R}_{k,\beta,n}(f)$ is defined almost everywhere. By using the relation (11) we deduce that for all $x \in \mathbb{R}^{d+1}_+$,

$$|{}^t\mathcal{R}_{k,\beta,n}(f)(x)| \leq M_0 e^{-a\|x\|^2},$$

where M_0 is a positive constant. So,

$$|{}^t\mathcal{R}_{k,\beta,n}(f)(x)| \leq M_0(1+|x_j|)^N e^{-a|x_j|^2}, j = 1, ..., d+1, \tag{25}$$

On the other hand from (11) and (24) we have for all $x \in \mathbb{R}^{d+1}_+$,

$$|\mathcal{F}_0({}^t\mathcal{R}_{k,\beta,n})(f)(y)| \leq M(1+|y_j|)^N e^{-b|y_j|^2}, j = 1, ..., d+1, \tag{26}$$

The relations (25) and (26) show that the conditions of Proposition 3.2 of [4] are satisfied by the function ${}^t\mathcal{R}_{k,\beta,n}(f)$. Thus we get:
i) If $ab > \frac{1}{4}$, ${}^t\mathcal{R}_{k,\beta,n}(f) = 0$ a.e. Using (11) we deduce

$$\forall y \in \mathbb{R}^{d+1}_+, \mathcal{F}_{k,\beta,n}(f)(y) = \mathcal{F}_0 \circ ({}^t\mathcal{R}_{k,\beta,n})(f)(y) = 0.$$

Then from Theorem 2.3.1 of [8] we have $f = 0$ a.e.
ii) If $ab = \frac{1}{4}$, then ${}^t\mathcal{R}_{k,\beta,n}(f)(x) = P(x)|e^{-a\|x\|^2}$, where P is a polynomial of degree strictly lower than N. The same proof as the end of theorem shows that

$$f(x) = \sum_{|s|+p\leq N} a^{k,\beta}_{s,p} W^{k,\beta}_{s,p}(\frac{1}{4a}, x)\ a.e.$$

iii) If $ab < \frac{1}{4}$, let $t\ \in]a, \frac{1}{4b}[$ and $f(x) = Ce^{-t\|x\|^2}$ for some real constant C, these functions satisfy the conditions (24).
■

5. COWLING-PRICE THEOREM FOR THE DUNKL-BESSEL TYPE TRANSFORM

Theorem 5..1 *(Cowling-Price type) Let $N \in \mathbb{N}$ and assume that $f \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ is such that*

$$\int_{\mathbb{R}^{d+1}_+} e^{a\|x\|^2}|f(x)|d\mu_{k,\beta+n}(x) < +\infty,\ and\ \int_{\mathbb{R}^{d+1}_+} \frac{e^{b\|y\|^2}}{(1+\|y\|)^N}|\mathcal{F}_{k,\beta,n}(f)|dy < +\infty \tag{27}$$

for some constants $a > 0$, $b > 0$. Then
i) If $ab > \frac{1}{4}$, we have $f = 0$ a.e.
ii) If $ab = \frac{1}{4}$, then when $N \geq d+2$ we have

$$f(x) = \sum_{|s|+p\leq \frac{N-d-1}{2}} a^{k,\beta}_{s,p} W^{k,\beta}_{s,p}(\frac{1}{4a}, x)\ a.e\ where\ a^{k,\beta}_{s,p} \in \mathbb{C}.$$

iii) If $ab < \frac{1}{4}$, there are infinity many nonzero functions f satisfying the condition (27).

Proof. From the first condition of (27) we deduce that $f \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$. So by Theorem 3, the function ${}^t\mathcal{R}_{k,\beta,n}(f)$ is defined almost everywhere. By using the relation (5), (8) and (27) we have:

$$\int_{\mathbb{R}^{d+1}_+} \frac{|{}^t\mathcal{R}_{k,\beta,n}(f)(x)|e^{a\|x\|^2}}{(1+\|x\|)^N}dx \leq \int_{\mathbb{R}^{d+1}_+} {}^t\mathcal{R}_{k,\beta,n}(e^{a\|x\|^2}|f|)(x)dx,$$

$$\leq \int_{\mathbb{R}^{d+1}_+} e^{a\|y\|^2}|f(y)|d\mu_{k,\beta+n}(y) < +\infty.$$

So

$$\int_{\mathbb{R}_+^{d+1}} \frac{|{}^t\mathcal{R}_{k,\beta,n}(f)(x)|e^{a\|x\|^2}}{(1+\|x\|)^N}dx < +\infty. \quad (28)$$

On the other hand from (11) and (27) we have:

$$\int_{\mathbb{R}_+^{d+1}} \frac{e^{b\|y\|^2}}{(1+\|y\|)^N}|\mathcal{F}_{k,\beta,n}(f)|dy = \int_{\mathbb{R}_+^{d+1}} \frac{e^{b\|y\|^2}}{(1+\|y\|)^N}|\mathcal{F}_0({}^t\mathcal{R}_{k,\beta,n})(f)(y)|dy < +\infty. \quad (29)$$

The relations (28) and (29) are the conditions of Proposition 3.2 of [2] which are satisfied by the function ${}^t\mathcal{R}_{k,\beta,n}(f)$. Thus we get: i) If $ab > \frac{1}{4}$, ${}^t\mathcal{R}_{k,\beta,n}(f) = 0$ a.e.
Using the same proof as of Theorem 5 we deduce $f(x) = 0$. a.e. $x \in \mathbb{R}_+^{d+1}$.
ii) If $ab = \frac{1}{4}$, then ${}^t\mathcal{R}_{k,\beta,n}(f)(x) = P(x)|e^{-a\|x\|^2}$, where P is a polynomial of degree strictly lower than $\frac{N-d-1}{2}$. The same proof as the end of theorem shows that

$$f(x) = \sum_{|s|+p\leq \frac{N-d-1}{2}} a_{s,p}^{k,\beta} W_{s,p}^{k,\beta}(\frac{1}{4a}, x)\ a.e.$$

iii) If $ab < \frac{1}{4}$, let $t \in]a, \frac{1}{4b}[$ and $f(x) = Ce^{-t\|x\|^2}$ for some real constant C, these functions satisfy the conditions (27). This complete the proof. ■

6. MORGAN TYPE FOR THE DUNKL-BESSEL TYPE TRANSFORM

Theorem 6..1 *(Morgan type) Let $1 < p < 2$ and q be the conjugate exponent of p. Assume that $f \in L^2_{k,\beta}(\mathbb{R}_+^{d+1})$ satisfies*

$$\int_{\mathbb{R}_+^{d+1}} e^{\frac{a^p}{p}\|x\|^p}|f(x)|d\mu_{k,\beta+n}(x) < +\infty,\ and \int_{\mathbb{R}_+^{d+1}} e^{\frac{b^q}{q}\|y\|^q}|\mathcal{F}_{k,\beta,n}(f)(y)|dy < +\infty, \quad (30)$$

for some constants $a > 0$, $b > 0$.
Then if $ab > |\cos(\frac{p\pi}{2})|^{\frac{1}{p}}$, we have $f = 0$ a.e.

Proof. From the first condition of (30) implies that $f \in L^1_{k,\beta}(\mathbb{R}_+^{d+1})$. So by Theorem 2, the function ${}^tR_{k,\beta}(f)$ is defined almost everywhere. By using the relation (5), (30) we have:

$$\int_{\mathbb{R}_+^{d+1}} |{}^t\mathcal{R}_{k,\beta,n}(f)(x)|e^{\frac{a^p}{p}\|x\|^p}dx \leq \int_{\mathbb{R}_+^{d+1}} e^{\frac{a^p}{p}\|y\|^p}|f(y)|d\mu_{k,\beta+n} < +\infty.$$

So

$$\int_{\mathbb{R}_+^{d+1}} |{}^t\mathcal{R}_{k,\beta,n}(f)(x)|e^{\frac{a^p}{p}\|x\|^p}dx < +\infty \quad (31)$$

On the other hand, from (11) and (30) we have:

$$\int_{\mathbb{R}_+^{d+1}} e^{\frac{b^q}{q}\|y\|^q}|\mathcal{F}_{k,\beta,n}(f)(y)|dy = \int_{\mathbb{R}_+^{d+1}} e^{\frac{b^q}{q}\|y\|^q}|\mathcal{F}_0({}^t\mathcal{R}_{k,\beta,n})(f)(y)|dy < +\infty. \quad (32)$$

The relations (31) and (32) are the conditions of Theorem 1.4 of [4] , which are satisfied by the function ${}^t\mathcal{R}_{k,\beta,n}(f)$. Thus we deduce that if $ab > |\cos(\frac{p\pi}{2})|^{\frac{1}{p}}$ we have ${}^t\mathcal{R}_{k,\beta,n}(f) = 0$ a.e. Using the same proof as the Theorem 5 we obtain $f(y) = 0$. a.e. $y \in \mathbb{R}_+^{d+1}$. ■

REFERENCES

[1] A. Abouelaz, A. Achak, R. Daher, El. Loualid, Harmonic analysis associated with the Dunkl-Bessel type-Laplace operator, (IJAREM) volume. 01, Issue 04, July 2015.

[2] A. Beurling, In: The Collected Works of Arne Beurling. Birkhuser, Boston (1989), 1-2.

[3] A. Bonami, B. Demange and P. Jaming, Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms. Rev. Mat. Iberoamericana 19 (2002), 22-35.

[4] L. Bouattour and K. Trimche, An analogue of Beurling-Hrmader's theorem for the Chbli-Trimche transfrom. Global J. of Pure and Appl. Math. 1, No 3(2005), 342-357.

[5] M. G. Cowling and J. F. Prices, Generalizations of Heisemberg's inequality. In: Lecture Notes in Math. 992, Springer, Berlin (1983), 443-449.

[6] G. H. Hardy, A theorem concerning Fourier transform. J. London Math. Soc. 8 (1933), 227-231.

[7] L. Hrmander, A uniqueness theorem of Beurling for Fourier transform pairs. Ark. For Math. 2, No 2 (1991), 237-240.

[8] H. Mejjaoli and K. Trimche, An analogue of Hardy's theorem and its L^p- version for the Dunkl-Bessel transform. Concrete Mathematica (2004).

[9] G. W. Morgan, A note on Fourier transforms. J. London Math. Soc. 9 (1934), 188-192.