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# Nonlinear Evolution equations : A Brief Review

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**ABSTRACT**

A brief review of nonlinear systems was undertaken including its brief history and an overview was given of some of the techniques used, like the Lax method and the AKNS method, to uncover solitary wave solutions and the specialized ones like the solitons.



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## 1. INTRODUCTION

Quite a large number of real-world physical systems are controlled by nonlinear partial differential equations (PDEs). PDEs in the context of nonlinear evolution equations (NLEEs) have been intensely studied during the past several decades [1, 2, 3, 4, 5]. This has led to the outpourings of new methods to tackle them either analytically or making use of numerical techniques. The propagation of wave motion requires an investigation of its various properties such as the surface height of a water wave, magnitude of an electromagnetic wave, presence of different classes of waves addressing, in particular, the cnoidal and solitary waves, and of specialized types like the soliton. The soliton is a localized wave [3,4,6] whose amplitude, shape and velocity are preserved even followung a collision with another wave. In other words, a remarkable quality of the soliton is that when one solitary wave collides with another its profile emerges unchanged after the collision except possibly for a phase shift. Solitary waves arise in both continuous systems as in the KdV equation and discrete cases such as the Toda lattice, and also in problems of multiple spatial dimensions. The envelop of the wave has one global peak and tends to decay far away from the peak. Solitary waves arise in many situations pertaining to the elevation of the surface of water and for the intensity of light in optical fibers.

In the literature, the words "soliton" and "solitary wave" have often been used interchangeably. The particle-like behaviour of stable solitary wave solutions led Zabusky and Kruskal [7] to coin the word "soliton". The investigation of physical phenomena has often focused on specialized systems that are mathematically tractable. Here is one is concerned with the feature of integrability which is a mathematical property that can be successfully utilized to extract quantitative information of the system character (like the conserved quantities it possesses) in relation to its dynamics both at the local and global level. Other significant properties of integrable systems are their universal nature, including how they can be effectively linearized through the techniques of inverse scattering transform (IST) or by employing other suitable methods like the Lax pair [8], Hirota's bilinear method for finding one soliton or multi solitons for integrable nonlinear PDEs [9], or by reduction to an appropriate Riemann-Hilbert problem. To mention other methods which have successfully dealt with nonlinear evolution equations are the pseudo spectral method, inverse scattering transformation method, homogeneous balance method [10, 11], Backlund transformation method [12, 13], truncated Painleve expansion [14], and several others like the tanh method [15, 16], exponential-function methods [17], Jacobi elliptic function method [18,19]. Without the application of these methods many equations would have remained unexplored. In particular, the inverse scattering transform technique (IST) [4] and the AKNS formalism [20, 21, 22] were developed to search for the rich class of soliton solutions pertaining to the NLEEs and, in particular, to the KdV equation, modified KdV equation, nonlinear Schrödinger equation (NLS) and sine-Gordon equation.

In 1972, Zakharov and Shabat (ZS) in a remarkable work [23] established that the NLS is not only solvable but possesses soliton solutions. In 1974, Ablowitz, Kaup, Newell and Segur (AKNS) [20], generalised the ZS method to demonstrate that the IST could be applicable to many evolution equations. In 1979, Gupta [21] derived a nonlinear Schrödinger equation by a simple extension of the Lax integrability criterion [8]. A multi-Hamiltonian structure is a typical feature with $(1 + 1)$-dimentional Lax integrable system along with infinitely many conservation laws and infinitely many symmetries. The class of travelling wave solutions plays a crucial role in the investigation of nonlinear physical phenomena in various fields.

For the linear system construction of new solutions by superposition of the known ones is a known advantage and is particularly important in the applications to the solutions of boundary problems. However, for the NLEEs there is no analogue of superposition principle until the solutions corresponding to the solitons were discovered. The existence of multi-soliton solutions can be regarded as asymptotic (nonlinear) superposition of separated solitary waves. Existence of soliton solutions and their corresponding Hamiltonians has radically changed the direction of research in the areas of classical mechanics, quantum mechanics and their allied fields including the employment of the group theory analysis. Solitons and solitary wave solutions are found to appear in almost all branches of physics such as, for instance, in the theories of hydrodynamics, plasma physics, nonlinear optics, condensed matter physics, nuclear physics, particle physics, low temperature physics, biophysics and astrophysics [1, 3, 4, 6, 24, 25, 26, 27, 28].

It is worth mentioning that the observation of a solitary wave was first documented by the Scottish engineer and naval architect John Scott Russell while making observations of the motion of his boat in the Edinburgh-Glasgow canal in 1834. The solitary wave was later interpreted to be basically a solution of a weakly nonlinear shallow water wave equation [29] by Korteweg and de Vries [30] in their landmark work published in 1895. Russell called his observational wave as the great wave of translation. He reported his findings as a Report on waves in 1844 [31]. The following lines of his paper have now become famous:

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on a rate of eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

Let us comment here that the KdV is one of the simplest nonlinear evolution equations [4, 2]. Its form reads

$$u_t + \alpha u u_x + \beta u_{xxx} = 0, \tag{1}$$

where $\alpha$ and $\beta$ are real constants. Its modified partner namely, the mKdV[4,2]

$$v_t + \alpha_1 v^2 v_x + \beta_1 v_{xxx} = 0, \tag{2}$$

where $\alpha_1$ and $\beta_1$ are real constants is related to the KdV by the Miura map [32] through the transformation

$$u = v^2 + v_x \quad (3)$$

KdV and mKdV equations form integrable systems. Their properties are summarized in [33] which also gives a coverage of their properties, the relevant recursion relations, the underlying Hamiltonians, symplectic and co-symplectic operators, roots of their symmetries and the scaling symmetry.

Some remarks on other modern methods developed during the last few years are in order:

In 1987, Kumar [34] presented an application of the isospectral Hamiltonian approach in soliton physics to generate a class of soliton profile. In 1992, Malfliet [15] suggested the use of the tanh-method to derive analytical soltions of the travelling wave type of a typical nonlinear evolution equations. In 1996, the method was improved by applying boundary conditions in the series expansion [35]. Subsequently, the travelling wave solutions of the KdV-Burgers equation, the KdV -mKdV equation, the dissipative dispersion equation, the extended mKdV-KdV-Burgers equation, the Fisher equation and the foam drainage equation were constructed to justify the efficiency of the method. In 1995, Wang [36] carried out a homogeneous balance method to extract the solitary wave solutions of a variant of Boussinesq equation. In 1996, Yan [37] presented sine-Gordon expansion method to provide a link between linear and nonlinear wave theories including the sine-Gordon equation. The applicability was made on the KdV, mKdV and regularized long wave equations. In 2001, Liu et al. [38] put forward the Jacobi elliptic function method to seek periodic wave solutions of the KdV equation, mKdV equation, Boussinesq equation, nonlinear KleinGordon equation and the coupled variant Boussinesq equation. In 2002, Feng [39] discovered the first integral method on the basis of the ring theory of commutative algebra. He illustrated the exact solutions of the nonlinear evolution equations by calculating a first integral through the division theorem. In 2004, Wazwaz [40] applied the sine-cosine method to get the travelling wave solutions of some well-known nonlinear models including the KdV equation, the modified KdV equation, the generalized KdV equation, Boussinesq equation, Benjamin-Bona-Mahony equation and $\phi^4$-equation. In 2005, Raju et al. [41] constructed exact solitary wave solutions of the NLS equation having a source term. In 2006, Prasad [42] studied amplitude death in coupled chaotic oscillators. In 2006, Pradhan and Panigrahi [43] studied the effect of the variable coefficients on the solution space of the mKdV equation in connection with the exact cnoidal wave and localize soliton solutions. They found out that the effect of distributed coefficients on the soliton dynamics of the mKdV equation is different from that of the nonlinear Schrödinger equation. In 2008, Wang et al. [44] considered a direct method, named $\left(\frac{G'}{G}\right)$ expansion method, and proved that it is an efficient scheme for obtaining exact travelling wave solutions of nonlinear evolution equations. In 2008, Zhang et al. [45] extended the ( $\frac{G'}{G}$ )expansion method to the nonlinear evolution equations with variable coefficients. In 2009, Saied et al. [46] proposed a direct and generalized algebraic method to extract establish more general exact solutions in terms of the Weierstrass elliptic function are obtained. In 2009, Biswas [47] obtained the solitary wave solution of the KdV equation in the presence of power-law nonlinearity and having linear damping and dispersion with time-dependent coefficients. In 2010, Bhaumik and Choudhury [48] proposed a novel model of chaotic dynamics as complemented shift map thereby proving some strong chaotic properties of the same map. In 2010, Bagchi et al. [49] dealt with an extended elliptic function method to obtain exact travelling wave solutions expressible in terms of quasi-periodic elliptic integral function and doubly-periodic Jacobian elliptic functions. In 2010, Zhang et al. [50] developed an improvement of the ( $\frac{G'}{G}$) expansion method to construct travelling wave solutions of the Zakharov-Kuznetsov-BBM equation and (2 + 1)-dimensional dispersive long wave equation. In 2011, Ma et al. [51] gave a simple direct method for finding exact solutions of different types of nonlinear evolution equations with variable coefficients. In 2012, Gupta et al. [52] explored a class of nonlinearity control parameter for bright solitons of non-autonomous NLS equation with a trapping potential. Kumar et al. [53] also showed the existence of of self-similar giant rogue waves in nonlinear fiber optics. In 2012, Biswas et al. [54] employed the solitary wave ansatz method to express the solitary wave solutions of Bona-Chen equation. In 2015, Mirzazadeh et al. [55] put forward the soliton solutions of optical couplers controlled by the Bernoulli's equation and employing the sine-cosine method. In 2015, Loomba et al. [56] investigated the bright solitons of the nonautonomous cubic-quintic nonlinear Schrodinger equation with sign-reversal nonlinearity. In 2016, Dudkowski et al. [57] analyzed and reviewed hidden attractors in dynamical systems. In 2018, Choudhury et al. [58] discussed cosmological aspects of some dynamical aspects of interacting quintessence model.

The arrival of high-speed electronic computers along with the advent of mathematical software and also the availability of well-structured analytical tools has incited a great amount of research in nonlinear dynamics. Exact solutions of NLEEs, when available, help to assist in the authentication of numerical treatments and stability analysis of the solutions. These have also triggered initiation of new concepts like the rogue waves [59], vortices [60], dispersion-managed solitons [61], super-continuum generation [62], modulation instability [63], complete integrability [2] etc.

## 2. Soliton solutions of the Korteweg-de Vries equation

The KdV equation appears in shallow water wave problems and in several class of other situations. It is also the governing equation for a string in the Fermi-Pasta-UlamTsingou (FPUT) problem [64] in the continuum limit. The parameters $\alpha$ and $\beta$ can be adjusted to express in the following form

$$u_t + 6uu_x + u_{xxx} = 0, \tag{4}$$

where $u(x,t)$ denotes the elongation of the wave at the location $x$ and time $t$, and the subscripts denote partial differentiation. The above form is how the KdV equation usually appears in the literature. The third term which is of third order in spatial derivative is the dispersive term responsible for the spreading of waves while the second term is the the nonlinear term which steepen the waves. The KdV equation admits solutions which are in general expressible in terms of elliptic functions. The soliton solution is the outcome of a balance between nonlinear and dispersive effects.

To find localized solitary wave solutions which are rapidly decreasing towards 0 as $|x| \to \pm\infty$, we let $u = l(x-ct) = l(\xi)$ for some constant $c > 0$ so that the solution represents a right-travelling wave. Hence $l$ satisfies the following ordinary differential equation

$$-cl' + 6ll' + l''' = 0 \tag{5}$$

where $' \equiv \frac{d}{d\xi}$. Integrating once gives

$$-cl + 3l^2 + l'' = C_1, \tag{6}$$

where $C_1$ is an arbitrary integrating constant. Multiplying both sides by $2l'$ and integrating once again yields

$$-cl^2 + 2l^3 + (l')^2 = C_1 l + C_2 \tag{7}$$

where $C_2$ is an arbitrary integrating constant. Recall that we are looking for a solution that characterizes a solitary wave meaning that away from the heap of water, there is no elevation. Mathematically, we seek localized solitary wave solutions. These would

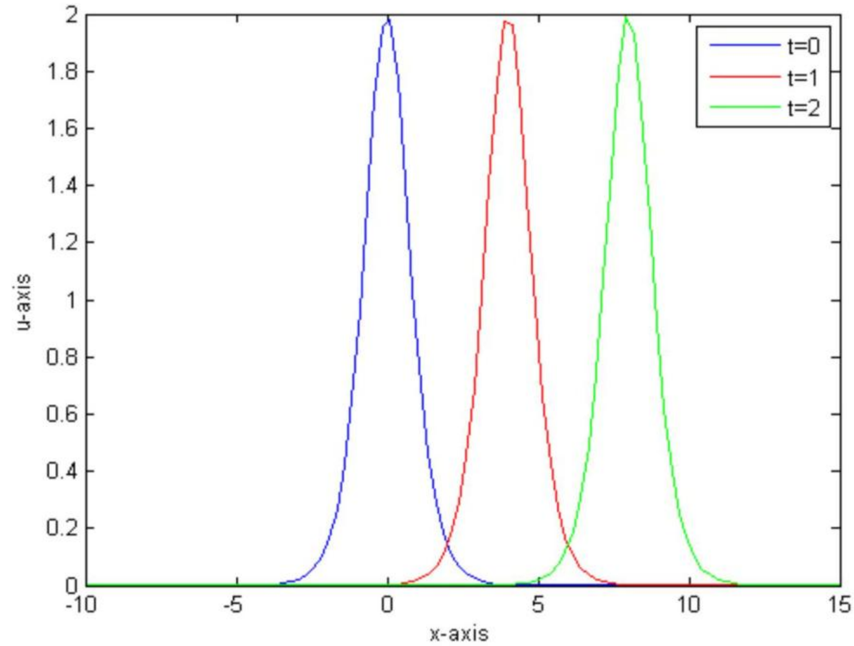


**Figure 1:** 1-soliton for different values of t.

correspond to the solitonic profile. This means $l(\xi), l'(\xi)$ and $l''(\xi)$ have to tend to zero as $\xi \to \pm\infty$ implying that we need to set the constants $C_1 = C_2 = 0.$ [1] Thus (7) can be written as

$$\left(\frac{dl}{d\xi}\right)^2 = l(cl - 2l^2). \tag{8}$$

By separation of variable, we may write

$$\int \frac{dl}{l\sqrt{(c - 2l)}} = \int d\xi \tag{9}$$

By making the substitution $l = \frac{c^2}{2}\theta$, equation (9) leads to the solution

$$l(\xi) = \frac{c^2}{2}\left(\frac{\sqrt{c}}{2}(\xi - x_0)\right), \tag{10}$$

where $x_0$ is the constant of integration. Evidently, it decays to zero exponentially as $x \to \pm\infty$. With the form (10), we arrive at the solution

$$u(x,t) = \frac{c^2}{2}\left(\frac{\sqrt{c}}{2}(x - ct - x_0)\right). \tag{11}$$

This is the one-soliton solution of the KdV equation. Since $^2\theta$ is an even function, $x_0$ can be termed the phase shift. When $c = 4$ and $x_0 = 0$, we obtain the particular solution,

$$u(x,t) = 2^2(x - 4t). \tag{12}$$

We illustrate graphically, the 1 -soliton solution of the KdV equation. Figure 1 depicts the 1 -soliton solution (12) with fixed shape and size for different values of $t$. We know that, no solution is possible for the sum of two solutions of the KdV equation since the KdV is nonlinear. However, we can extract a solution relevant to an arbitrary number of solitary waves or 1 -solitons of different sizes. Such a class of solutions is referred as the multi-soliton or $n$-soliton solutions.

**3. The Lax method**

In 1968, Peter Lax introduced a tool [8] for finding conserved quantities of some evolutionary differential equations. Lax pair comprises two operators, $L$ and $M$ that act on elements of $L^2(\mathbb{R})$, the space of integrable functions on the real line, endowed with an inner product

$$\langle \phi, \psi \rangle = \int_{-\infty}^{+\infty} \phi\psi dx. \tag{13}$$

Both $L$ and $M$ are self-adjoint so that $\langle L[\phi,\psi]\rangle = \langle \phi, L[\psi]\rangle$ and $\langle M[\phi,\psi]\rangle = \langle \phi, M[\psi]\rangle$ $\forall \phi, \psi \in L^2(\mathbb{R})$. To find exact solutions via inverse scattering, the spectral problem

$$L[\psi] = \lambda\psi, \tag{14}$$

is to be addressed so that $\psi$ is an eigenfunction for $L$ with eigenvalue $\lambda$. The eigenfunction $\psi$ evolves in time in the manner

$$\psi_t = M[\psi]. \tag{15}$$

Lax showed that if equations (14) and (15) both holds, then the operators $L$ and $M$ satisfy the relation

$$L_t + [L, M] = 0, \tag{16}$$

where $[L, M] = LM - ML$ denotes the commutator of $L$ and $M$. Now, differentiating both sides of (14) with respect to $t$ and then using equation (15) we obtain

$$\begin{aligned} \lambda_t \psi &= L_t[\psi] + L[\psi_t] - \lambda \psi_t \\ &= L_t[\psi] + LM[\psi] - M[\lambda\psi] \\ &= L_t[\psi] + LM[\psi] - ML[\psi] \end{aligned} \tag{17}$$

Solving equation (17) for nontrivial eigenfunction $\psi$ and choosing $\lambda_t = 0$ yields equation (16). This also implies that every eigenvalue of $L$ is a constant

The KdV equation (4) provides us with a prototypical example of the Lax representation. If we consider the Lax pair in terms of the following differential operators

$$L = \frac{\partial^2}{\partial x^2} + u \tag{18}$$

then

$$[L, M][\psi] = (u_{xxx} + 6uu_x)\psi. \tag{19}$$

Substituting equation (19) into the Lax equation (16), we obtain

$$\frac{\partial L}{\partial t} = \frac{\partial u}{\partial t} = -(u_{xxx} + 6uu_x), \tag{20}$$

which is the familiar KdV equation (4). Thus, we see that $L$ and $M$ satisfy Lax equation (16) and that $u$ obeys the KdV equation. So, it is clearly established, here, that when a nonlinear evolution equation is represented in the aforesaid manner it is supposed to possess Lax representation and the two operators involved are referred to as a Lax pair.

Not only the KdV, but mKdV equation which reads

$$v_t - 6v^2 v_x + v_{xxx} = 0, \tag{21}$$

also exhibits a Lax representation. The explicit representations of $L$ and $M$ are

$$L = \frac{\partial^2}{\partial x^2} - (v^2 + v_x) \tag{22}$$

In 1968, Miura [32] pointed out that if $v$ satisfies equation (21), then the quantity $u$ represented by

$$u = -(v^2 + v_x), \tag{23}$$

satisfies the KdV equation (4). In fact, substituting (23) into (4) one finds

$$2v(v_t - 6v^2 v_x + v_{xxx}) + (v_t - 6v^2 v_x + v_{xxx})_x = 0, \tag{24}$$

which readily confirms compatibility with (21).

## 4. AKNS Formalism

In 1967, Kruskal and his coworkers [65] applied the IST technique to the KdV equation whereby the KdV equation is transformed to a linear Schrödinger spectral problem. Later, in 1972, ZS [23] showed that the nonlinear Schrödinger equation of the type

$$iu_t + u_{xx} + u|u|^2 = 0, \tag{25}$$

is also exactly solvable through IST that exhibits soliton like solutions. This turned out to be a noteworthy development in this direction. This extension was actually accomplished by associating the NLS equation with a Lax pair which could be represented by $2 \times 2$ matrix linear differential operators. It revealed that the IST technique is not at all restricted to the KdV equation but indeed could be applicable to a much wider sphere. What is more significant is that the analysis of ZS confirmed that the solitonic property is a more encompassing phenomenon possessed by a class of nonlinear dispersive systems. It motivated researchers to seek general prescriptions to identify wider class of integrable systems.

In 1974, Ablowitz, Kaup, Newell and Segur (AKNS) [20], published a paper that established the applicability of IST to a greater number of evolution equations. The AKNS scheme starts from a generalisation of the Sturm-Liouville equation by regarding the latter as a pair of first order equations. This generates a $2 \times 2$ eigenvalue problem. Both the AKNS and ZS schemes are applicable to the KdV and NLS equations as well as to the mKdV and sine-Gordon (SG) equations. In the following, we quickly describe how the scheme of AKNS works.

The linear eigenvalue problem is given by [4, 21, 20]

$$L[u]\psi(x,t) = \lambda\psi(x,t), \tag{26}$$

where $\lambda$ is the eigenvalue and $\psi(x,t)$ evolve with time as given by

$$\partial_t\psi(x,t) = A[u]\psi(x,t). \tag{27}$$

Consider the eigenvalue problem for (26)

$$\psi_x = M\psi;\ \psi = \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix};\ M = \begin{bmatrix}-i\xi & q(x,t)\\ r(x,t) & i\xi\end{bmatrix}, \tag{28}$$

$$\text{i.e. } \begin{bmatrix}\frac{\partial}{\partial x} & -q(x,t)\\ r(x,t) & -\frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} = -i\xi\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}, \tag{29}$$

where $q \equiv q(x,t)$ and $r \equiv r(x,t)$ are potentials, $\xi$ is the eigenvalue and the time dependence of the solution is taken according to (27):

$$\psi_t = N\psi;\ N = \begin{bmatrix}A & B\\ C & D\end{bmatrix}. \tag{30}$$

The compatibility condition $\psi_{xt} = \psi_{tx}$ reads

$$N_x - M_t + [N,M] = \mathbf{0}, \tag{31}$$

where $[N,M] = NM - MN$. This immediately leads to the equations

$$\begin{aligned} A_x &= qC - rB - i\xi_t,\\ B_x &= -2i\xi B + q_t - 2qA,\\ C_x &= 2i\xi C + r_t + 2rA, \end{aligned} \tag{32}$$

In the above we assumed that the nature of the eigenvalue parameter is isospectral. In other words, it does not change with time.

We can generate meaningful evoluation equations from the master equations (32). Note that the eigenvalue parameter $\xi$ appears explicitly. As a result, we can take the quantities $A, B, C$ and $D$ to be well defined functions of $\xi$ so that they can be expanded in a powers series with respect to $\xi$ or $\xi^{-1}$. Truncating the power series suitably so that (32) remain consistent, one can obtain different types of nonlinear evolution equations.

### 5. Summary

To summarize, a review of nonlinear systems was presented, including a brief historical background. Key analytical techniques, such as the Lax method and the AKNS method, were discussed for deriving solitary wave solutions, with particular emphasis on solitons.

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